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MAS

- AUTONOMNÍ AGENTI A MULTIAGENTNÍ SYSTÉMY
- NEKOOPERATIVNÍ TEORIE HER

- COMPONENTS OF AGENT ARCHITECTURES
  - ACTIONS
  - PERCEPTS
  - DECISION MAKING
    - $d: P^* \rightarrow A$
    - MAPS PERCEPTION HISTORY TO ACTIONS

- TYPES OF ARCHYTECTURES
  - 1. REFLEX (REACTIVE) AGENTS
  - 2. MODEL-BASED REFLEX AGENT
  - 3. MODEL-BASED GOAL-BASED AGENT
  - 4. MODEL-BASED UTILITY-BASED AGENT
  - 5. LEARNING BASED AGENT

- REFLEX AGENT
  - DECIDES SOLELY ON CURRENT PERCEPTS

- MODEL-BASED REFLEX AGENT
  - AGENT PERCEPTS AND GRADUALY BUILDS MODEL OF ENVIRONMENT
  - DECISION BASED ON THE MODEL OF ENVIRONMENT

- MODEL-BASED GOAL-BASED AGENT
    - ACTIONS ARE CHOSEN TO REACH DECLARATIVELY SPECIFIED GOAL
    - MULTIPLE TECHNIQUES
        - PLANNING
        - BELIEF-DESIRE-INTENTION ARCHITECTURE
- MODEL-BASED UTILITY-BASED AGENT
    - SOME WAYS TO ACHIEVE GOALS ARE PREFERED
    - TECHNIQUES
        - NON-ADVERSARIAL
            - MDP
            - PO MDP
        - ADVERSARIAL
            - SEQUENTIAL GAMES
- LEARNING BASED AGENT
    - AGENT DOES NOT KNOW THE TASK FULLY
        - WHAT ACTIONS DOES
        - WHAT IS THE GOAL
    - HE LEARNS THE TASK
    - HE LEARNS BOTH MODEL AND STRATEGY

~~- BELIEF-DESIRE-INTENTION~~
    ~~- BDI~~
    - FOR ~~MODEL-BASED GOAL-BASED AGENTS~~
    ~~- BELIEFS~~

- MULTI AGENT SYSTEM
    - COLLECTION OF AUTONOMOUS AGENTS
    - EACH ACTING TOWARDS ITS OBJECTIVE
    - AGENTS INTERACTS IN SHARED ENVIRONMENT
    - AGENTS ARE ABLE TO COMUNICATE AND COORDINATE THEIR ACTIONS
- PROPERTIES OF INTELIGENT AGENT
    - AUTONOMY
        - AGENT IS RESPONSIBLE FOR ITS INNER STATE
        - AGENT DECIDES INDIVIDUALLY ABOUT ITS ACTIONS
    - REACTIVITY
        - AGENT IS CAPABLE OF NEAR-REAL TIME DECISION WITH RESPECT TO CHANGES IN ENVIRONMENT OR SOCIL NEIGHBOURHOOD
    - INTENTIONALITY
        - AGENT HAS LONG TERM INTENTION
        - AGENT MEETS DESIGNER'S OBJECTIVES
    - RATIONALITY
        - AGENT IS CAPABLE OF INTELLIGENT RATIONAL DECISION MAKING
        - AGENT CAN MAXIMIZE HIS UTILITY
    - SOCIAL CAPABILITY
        - AGENT CAN INTERACT WITH OTHER AGENTS
- AGENT VS OBJECT
    - AGENT IS UNPREDICTABLE
    - ~~OBJECT~~ AGENT CONSISTS OF OBJECTS
    - OBJECT CAN CONSIST OF OTHER OBJECTS
    - AGENT CAN'T CONSIST OF OTHER AGENTS

- MICRO VS. MACRO PERSPECTIVE
  - MICRO
    - AGENT DESIGN PROBLEM
    - HOW SHOULD AGENTS ACT TO CARRY OUT THEIR TASKS?
  - MACRO
    - SOCIETY DESIGN PROBLEM
    - HOW SHOULD AGENTS INTERACT TO CARRY OUT THEIR TASKS?

- AGENT
  - ANYTHING THAT CAN PERCEIVE ITS ENVIRONMENT AND ACT ON THAT ENVIRONMENT
  - CAPABLE OF AUTONOMOUS ACTIONS IN ORDER TO MEET ~~B~~ ITS DESIGN GOALS

- RATIONAL BEHAVIOUR
  - RATIONAL AGENT CHOOSES ACTIONS WHICH MAXIMIZES ITS UTILITY GIVEN SEQUENCE OF PERCEPTS AND BUILD-IN KNOWLEDGE

- SPECIFICATION OF TASK ENVIRONMENT
  - PEAS
    - PERFORMANCE MEASURE
    - ENVIRONMENT
    - ACTUATORS
    - SENSORS

- PROPERTIES OF ENVIRONMENT
  - FULLY OBSERVABLE, PARTIALLY OBSERVABLE
  - DETERMINISTIC, STOCHASTIC
  - EPISODIC, SEQUENTIAL
  - STATIC, DYNAMIC
  - DISCRETE, CONTINUOUS
  - SINGLE AGENT, MULTI AGENT

- RATIONALITY

  - $a = \arg\max_{l \in L} \sum_{p_i : o_i \in l} p_i u(o_i)$

- BOUNDED RATIONALITY

  - CAPABILITY OF AGENT TO PERFORM RATIONAL DECISIONS GIVEN BOUNDS OF COMPUTATIONAL RESPONSES

    - TIME
    - MEMORY

- CALCULATIVE RATIONALITY

  - CAPABILITY TO PERFORM RATIONAL CHOICE EARLIER THAN FASTEST CHANGE IN ENVIRONMENT CAN OCCUR

- SELF INTERESTED RATIONAL AGENT

  - SELECT ACTIONS THAT OPTIMIZES ITS INDIVIDUAL UTILITY

    $a = \arg\max_{l \in L} \sum_{p_i : o_i \in l} p_i U(o_i)$

- COOPERATIVE RATIONAL AGENT

  - SELECT ACTIONS MAXIMIZING COLLECTIVE UTILITY

    $a = \arg\max_{l \in L} \sum_{\forall a_j \in A - a} \sum_{p_{i,j} : o_{i,j} \in l_j} p_{i,j}\, u(o_{i,j}) + \sum_{p_i : o_i \in l} p_i u(o_i)$

- POSSIBLE TASKS

  - COALITION FORMATION

    - FORMING TEAMS THAT HAVE HIGHEST VALUE

  - DISTRIBUTED COORDINATION

    - COORDINATING ASSIGNMENTS OF TASKS THAT CONSTRAINTS ARE MET AND OBJECTIVE FUNCTION MAXIMIZED

  - AUCTIONS

    - ALLOCATE SCARCE RESOURCE AND DETERMINE PAYMENTS THAT PROFIT IS MAXIMIZED

  - SOCIAL CHOICE

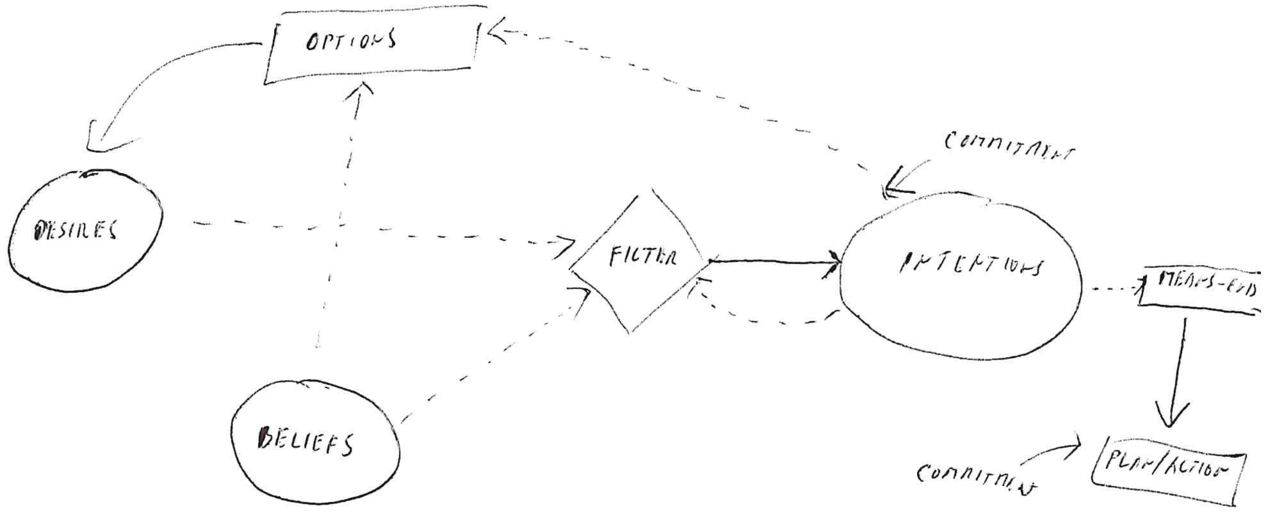
    - AGREE ON SINGLE CHOICE BETWEEN MULTIPLE AGENTS WITH DIFFERENT PREFERENCES

- THEORETICAL REASONING
  - REASONING DIRECTED TOWARDS BELIEFS
  - CONCERNED WITH DECIDING WHAT TO BELIEVE
  - PROCESS BY WHICH YOU CHANGE YOUR BELIEFS AND EXPECTATIONS

- PRACTICAL REASONING
  - DIRECTED TOWARDS ACTIONS
  - CONCERNED WITH DECIDING WHAT TO DO
  - DECIDES WHAT INDIVIDUALS SHOULD DO
  - PROCESS BY WHICH YOU CHANGE YOUR CHOICES, PLANS AND INTENTIONS
  - COMPONENTS
    - DELIBERATION
      - WHAT STATE OF AFFAIRS WE WANT TO ACHIEVE
      - CONSIDERING PREFERENCES, CHOOSING GOALS
      - OUTPUTS ARE INTENTIONS
    - MEANS - END REASONING
      - DECIDING HOW TO ACHIEVE SELECTED STATE OF AFFAIRS
      - BUILDING COURSES OF ACTION
      - OUTPUTS ARE PLANS

- DESIRES
  - STATES OF AFFAIRS THAT ARE CONSIDERED FOR ACHIEVEMENT
    - BASIC PREFERENCES
  - THEIR ARE WEAKER THAN INTENTS

- INTENTS
  - ~~WE~~ WE MAKE REASONABLE ATTEMPTS TO FULFILL INTENTIONS ONCE WE FORM THEM



- COMMITMENTS
  - AGREEMENT TO DO SOMETHING IN FUTURE
  - WAY TO STICK TO THE INTENT
  - IT IMPLIES TEMPORAL PERSISTENCE
  - AGENT HAS TO COMMIT TO BOTH INTENTS AND PLANS
  - DEGREES
    - BLIND/FANATICAL COMMITMENT
      - AGENT CONTINUES TO MAINTAIN INTENTION, UNTIL IT BELIEVES THE INTENTION HAS ACTUALLY BEEN ACHIEVED
    - SINGLE MINDED COMMITMENT
      - AGENT MAINTAINS INTENTION UNTIL IT BELIEVES THE INTENTION IS ACHIEVED OR IT IS NO LONGER POSSIBLE TO ACHIEVE IT
    - OPEN MINDED COMMITMENT
      - AGENT MAINTAINS INTENTION UNTIL ACHIEVED AS LONG AS IT IS STILL BELIEVED POSSIBLE

- BDI ARCHITECTURE
  - BELIEV, DESIRE, INTENTION
  - BELIEV
    - INFORMATION ABOUT WORLD
    - IT IS NOT KNOWLEDGE, AGENT CAN BELIEVE FACTS THAT ARE NOT TRUE
  - EVENTS
    - GOALS TO RESOLVE
    - CAN BE INTERNAL OR EXTERNAL
  - PLAN LIBRARY
    - RECIPES FOR HANDLING GOALS - EVENTS
  - INTENTIONS
    - PARTIALLY UNINSTANTIATED PROGRAMS WITH COMMITMENT
    - DETERMINED DYNAMICALLY BY AGENT AT RUNTIME
      - BASED ON KNOWN FACTS, GOALS AND AVAILABLE PLANS
    - IT REPRESENTS FOCUS OF ATTENTION
      - SOMETHING AGENT IS CURRENTLY WORKING ON
    - AGENT CAN HAVE SEVERAL INTENTIONS ACTIVE AT A TIME
    - NEW INTENTION IS CREATED FOR NEW EVENT ADDRESSING

- AGENTSPEAK LANGUAGE AND JASON INTERPRETER
  - PROLOG SYNTAX
  - EVENTS
    - +b - BELIEF ADDITION
    - -b - BELIEF DELETION
    - +!g - ACHIEVEMENT-GOAL ADDITION
    - -!g - ACHIEVEMENT-GOAL DELETION
    - +?g - TEST GOAL ADDITION
    - -?g - TEST GOAL DELETION

- PLANS ARE CONTEXT-SENSITIVE AND EVENT-INVOKED RECIPES
  - TRIGGERING EVENT : CONTEXT $\leftarrow$ PLAN BODY

    $+concert(A,V) : likes(A) \leftarrow\ !book\_tickets(A,V).$

- INTENTIONS ARE EXECUTED ONE AT A TIME

- SEMANTICS

  $\langle B, P, E, A, I, S_\varepsilon, S_O, S_I \rangle$

  - B IS SET OF BELIEFS
  - P IS SET OF PLANS
  - E IS SET OF EVENTS
  - A IS SET OF ACTIONS THAT CAN BE PERFORMED IN ENVIRONMENT
  - I IS SET OF INTENTIONS EACH OF WHICH IS STACK OF PARTIALLY INSTANTIATED PLANS
  - S_ε, S_O, S_I ARE SELECTION FUNCTIONS FOR EVENTS, OPTIONS, AND INTENTIONS

- MODAL LOGIC
  - TWO NEW CONNECTIVES
    - $\Box \varphi$
      - φ IS NECESSARILY TRUE
    - $\Diamond \varphi$
      - φ IS POSSIBLY TRUE
  - THIS HOLDS

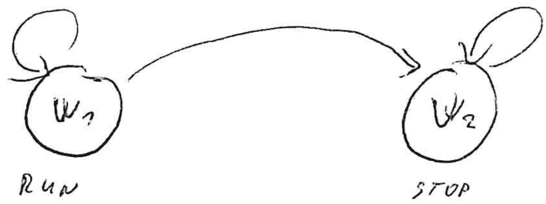
    $$\Diamond \varphi \leftrightarrow \neg \Box \neg \varphi$$

  - MODAL LANGUAGE
    - SMALLEST SET CONTAINING
      - ATOMIC PROPOSITIONS $p, q, r \ldots$
      - FOR FORMULAE φ, IT ALSO CONTAINS $\neg\varphi, \Box_1\varphi, \ldots, \Box_n\varphi$
      - FOR FORMULAE φ, ψ, IT ALSO CONTAINS $\varphi \wedge \psi$
    - WE TREAT $\vee, \rightarrow, \leftrightarrow, \Diamond$ AS MACROS DEFINED AS USUAL

- P IS NECESSARY ($\Box P$) $\Leftrightarrow$ P IS TRUE IN ALL POSSIBLE SCENARIOS
- P IS POSSIBLE ($\Diamond P$) $\Leftrightarrow$ P IS TRUE IN AT LEAST ONE POSSIBLE SCENARIO

- KRIPKE STRUCTURE
  - TOUPLE $\langle W, R \rangle$
    - W IS SET OF POSSIBLE WORDS
    - R IS BINARY RELATION ON WORDS CALLED ACCESSIBILITY RELATION
      - R INDICATES WHICH WORDS ARE RELEVANT FOR EACH OTHER
      - $w_1 R w_2$ ... "WORD 1 IS RELEVANT (REACHABLE) FROM WORLD w_2"
- KRIPKE MODEL
  - POSSIBLE WORLDS MODEL $M = \langle S, \pi \rangle$
    - S IS KRIPKE STRUCTURE
    - π IS VALUATION OF PROPOSITIONS

      $\pi : W \to P(\{p, q, r, \dots\})$
- TRUTH OF FORMULAE IS RELATIVE TO KRIPKE MODEL $M = \langle W, R, \pi \rangle$ AND WORLD $w \in W$.
  - $M, w \models p$ IFF $p \in \pi(w)$
  - $M, w \models \neg \varphi$ IFF NOT $M, w \models \varphi$
  - $M, w \models \varphi \wedge \psi$ IFF $M, w \models \varphi$ AND $M, w \models \psi$
  - $M, w \models \Box \varphi$ IFF FOR EVERY $w' \in W$ SUCH THAT $w R w'$ WE HAVE $M, w' \models \varphi$
- EXAMPLE



RUN $\to \Diamond$ STOP
STOP $\to \Box$ STOP
RUN $\to \Diamond \Box$ STOP

- MODAL LOGIC CAN BE TRANSLATED TO CLASSICAL LOGIC
  - BUT IT IS UGLY AND HARD TO AUTOMATE

- AXIOMS OF MODAL LOGIC
  - DISTRIBUTION AXIOM

$$K: (\Box\varphi \wedge \Box(\varphi \rightarrow \psi)) \rightarrow \Box\psi$$

  - GENERALIZATION AXIOM

$$\frac{\varphi}{\Box\varphi}$$

  - SYSTEM K IS SOUND AND COMPLETE WITH RESPECT TO CLASS OF ALL KRIPKE MODELS.

  - $T: \Box\varphi \rightarrow \varphi$
  - $D: \Box\varphi \rightarrow \Diamond\varphi$
  - $4: \Box\varphi \rightarrow \Box\Box\varphi$
  - $B: \varphi \rightarrow \Box\Diamond\varphi$
  - $5: \Diamond\varphi \rightarrow \Box\Diamond\varphi$

- BELIEF AND MODAL LOGIC
  - WE WANT
    - SATISFY K AXIOM
    - AGENT KNOW WHAT IT DOES KNOW (AXIOM 4)
    - AGENT KNOWS WHAT IT DOES NOT KNOW (AXIOM 5)
    - IT BELIEFS ARE NOT CONTRADICTORY, IF IT KNOWS SOMETHING, IT DOES NOT ALLOW THE NEGATION OF IT BEING TRUE (AXIOM D)
  - BELIEV IS KD45 SYSTEM

- KNOWLEDGE
  - EXTENSION OF BELIEF
  - THE KNOWLEDGE NEEDS TO BE TRUE (BELIEF NOT)
  - WE MUST ADD T AXIOM ($\Box\varphi \to \varphi$)
  - KNOWLEDGE IS KTD45 SYSTEM
- AUTOMATED REASONING
  - φ CAN BE TRUE IN M AND q ($M, q \models \varphi$)
  - φ CAN BE VALID IN M ($M, q \models \varphi$ FOR ALL q)
  - φ CAN BE VALID ($M, q \models \varphi$ FOR ALL M, q)
  - φ CAN BE SATISFIABLE ($M, q \models \varphi$ FOR SOME M, q)
  - φ CAN BE THEOREM
  - LOCAL MODEL CHECKING
    - GIVEN M, q AND φ
    - IS φ TRUE IN M, q ?
  - GLOBAL MODEL CHECKING
    - GIVEN M AND φ
    - WHAT IS SET OF STATES IN WHICH φ IS TRUE?
  - SATISFIABILITY
    - GIVEN φ
    - IS φ TRUE IN AT LEAST ONE MODEL AND STATE?
  - VALIDITY
    - GIVEN φ
    - IS φ TRUE IN ALL MODELS AND THEIR STATES
  - THEOREM PROVING
    - GIVEN φ IS IT POSSIBLE TO PROVE (DERIVE) φ?

- MODAL LOGIC IS GENERIC FRAMEWORK
  - WORK WITH
    - KNOWLEDGE
    - BELIEF
    - TIME
    - ACTION
    - ABILITY
    - ...

- GAME THEORY
  - STUDY OF STRATEGIC DECISION MAKING
  - OF MATHEMATICAL MODELS OF CONFLICT AND COOPERATION BETWEEN INTELIGENT RATIONAL DECISION MAKERS
  - GIVEN THE RULE OF GAME, GAME THEORY STUDIES STRATEGIC BEHAVIOUR OF THE AGENTS IN THE FORM OF STRATEGY
  - TYPES OF GAMES
    - COOPERATIVE, NON-COOPERATIVE
    - SYMMETRIC, ASYMMETRIC
    - ZERO SUM, NON ZERO SUM
    - SIMULTANEOUS, SEQUENTIAL
    - IMPERFECT INFORMATION GAMES
    - INFINITELY LONG GAMES
  - GAME DEFINITION
    - FINITE n PERSON GAME $\langle N, A, u \rangle$
      - N.. FINITE SET OF PLAYERS
      - $A = A_1 \times \ldots \times A_N$
        - A_i IS ACTION SET OF PLAYER i
        - $a \in A$ IS ACTION PROFILE, A IS SPACE OF ACTION PROFILES
      - $u = \langle u_1, \ldots, u_n \rangle$ IS UTILITY FUNCTION FOR EACH PLAYER
        - $u_i : A \to \mathbb{R}$

- STRATEGY s_i
  - DECISION AT EACH STATE OF THE GAME THAT AGENT i MAKES
- OUTCOME
  - SET OF POSSIBLE STATES RESULTING FROM AGENT'S DECISION MAKING
- STRATEGY PROFILE
  - SET OF STRATEGIES PLAYED BY AGENTS
  - $S = S_1 \times \ldots \times S_n$
- SOCIAL WELFARE
  - COLLECTIVE UTILITY

$$U(a) = \sum_{\forall i} u_i(a_i)$$

  - COOPERATIVE AGENTS TRYIES TO MAXIMIZE
  - SELF INTERESTED AGENTS TRYES TO MAXIMIZE ONLY ITS UTILITY
- PARETO EFFICIENCY
  - ACTION (STRATEGY)
  - IF THERE IS NO OTHER ACTION, THAT AT LEAST ONE AGENT IS BETTER OFF AND NO OTHER AGENT IS WORSE OFF THAN IN THE GIVEN PROFILE
- DOMINANCE
  - COMPARING TWO STRATEGIES
  - b DOMINATES WEAKLY a IF

$$a \preceq \text{ IFF } \forall i : u_i(a_i) \leq u_i(b_i)$$

  - DOMINANT STRATEGY IS NOT DOMINATED BY OTHER STRATEGY
  - PARETO EFFICIENT STRATEGY IS NOT WEAKLY DOMINATED BY OTHER STRATEGY

- NASH EQUILIBRIUM

  - BEST RESPONSE

    $$a_i^* \in BR(a_{-i}) \text{ iff } \forall a_i \in A_i,\ u_i(a_i^*, a_{-i}) \geq u_i(a_i, a_{-i})$$

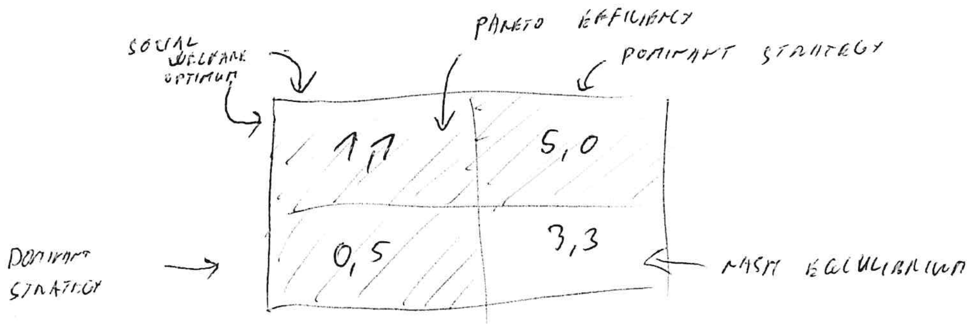
  - NASH EQUILIBRIUM

    - STRATEGY PROFILE ~~AS NASH EQUILIBRIUM~~ $a = \langle a_1, \dots, a_n \rangle$ IS IN NASH EQUILIBRIUM IFF $\forall i,\ a_i \in BR(a_{-i})$

  - STRONG NASH EQUILIBRIUM

    - IT IS STABLE AGAINST DEVIATIONS BY COOPERATION

  - PRISINORS DILEMA (~~[illegible]~~ MINIMALIZATION)



- MIXED STRATEGIES

  - $G = (N, A, u)$ IS NORMAL FORM GAME

  - SET OF MIXED STRATEGIES S_i FOR PLAYER i

    - SET OF PROBABILITY DISTRIBUTIONS

      $$A_i : S_i = \Delta(A_i)$$

  - PLAYER THAT SELECT PURE STRATEGY ACCORDING TO DISTRIBUTION

  - EXTENTION OF UTILITY FUNCTION TO EXPECTED UTILITY

    $$u_i(s) = \sum_{a \in A} u_i(a) \prod_{j \in N} s_j(a_j)$$

- DOMINANCE
  - STRONG
    - s_i STRONGLY DOMINATES s'_i
    - $\forall s_{-i} \in S_{-i}, \; u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$
  - WEAK
    - s_i WEAKLY DOMINATES s'_i
    - $\forall s_{-i} \in S_{-i}, \; u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$ AND $\exists s_{-i} \in S_{-i}, u_i(s_i, s_{-i}) > u_i($
    - "AT LEAST FOR ONE ENEMY'S ACTION THE STRATEGY IS BETTER THAN REST"
  - VERY WEAK
    - s_i VERY WEAKLY DOMINATES s'_i
    - $\forall s_{-i} \in S_{-i}, \; u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$

- NASH THEORY
  - EVERY GAME WITH FINITE NUMBER OF PLAYERS AND ACTION PROFILES HAS AT LEAST ONE NASH EQUILIBRIUM IN MIXED STRATEGIES

- SUPPORT OF MIXED STRATEGY
  - SET OF PURE STRATEGIES, WHICH HAS NON-ZERO PROBABILITY

    $\{ a_i \mid s_i(a_i) > 0 \}$

- EXAMPLE OF FINDING NASH EQUILIBRIUM

	L	R
U	2 1	0 0
D	0 0	1 2

FOR COLUMN PLAYER

$$E\,u_1(U) = E\,u_1(D)$$

$$2p + 0\cdot(1-p) = 0p \cdot 1(1-p)$$

$$p = \frac{1}{3}$$

$$L = \frac{1}{3} \qquad R = \frac{2}{3}$$

- MULTIPLE NASH EQUILIBRIUMS
  - WHICH ONE TO PLAY?
  - EQUILIBRIUM SELECTION PROBLEM
  - PLAYING NASH EQUILIBRIUM DOESN'T GIVE ANY GUARANTEE OF EXPECTED PAYOFF
  - GUARANTEES CAN BRING ~~GUAR~~ MAX MIN STRATEGIES
    - MAXMIN STRATEGY FOR PLAYER i

      $$\arg\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$$

    - VALUE OF MAXMIN FOR i

      $$\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$$

    - CONSERVATIVE STRATEGIES AGAINST WORST-CASE OPPONENTS
  - MIN MAX STRATEGIES
    - MINMAX STRATEGY FOR PLAYER i AGAINST $-i$

      $$\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$$

    - REPRESENT PUNISHMENT STRATEGIES FOR PLAYER $-i$

- MIN MAX THEOREM
  - IN FINITE, TWO PLAYER, ZERO-SUM GAME
  - IN ANY NASH EQUILIBRIUM
  - EACH PLAYER RECIEVES PAYOFF THAT IS EQUAL BOTH TO
    - HIS MAX MIN AND MIN MAX VALUE
  - CONSEQUENCES
    - WE CAN SAFELY PLAY NASH STRATEGIES IN ZERO SUM GAMES
    - ALL NASH EQUILIBRIA HAVE THE SAME PAYOFF IN TWO PLAYER ZERO SUM GAMES
      - MAXMIN OF PLAYER ONE IS CALLED VALUE OF THE GAME

- LP FOR NE IN ZERO-SUM TWO PLAYER GAME

$$\max_{s,U} U$$

$$\text{st.} \sum_{a_i \in A_1} s(a_1)\, u_1(a_1, a_2) \geq U \quad \forall a_2 \in A_2$$

$$\sum_{a_1 \in A_1} s(a_1) = 1$$

$$s(a_1) \geq 0 \quad \forall a_1 \in A_1$$

- CAN BE DONE IN POLYNOMIAL TIME

- LCP FOR NE IN GENERAL SUM GAMES

$$\sum_{a_2 \in A_2} u_1(a_1, a_2)\, s_2(a_2) + q(a_1) = U_1 \quad \forall a_1 \in A_1$$

$$\sum_{a_1 \in A_1} u_2(a_1, a_2)\, s_1(a_1) + w(a_2) = U_2 \quad \forall a_2 \in A_2$$

$$\sum_{a_1 \in A_1} s_1(a_1) = 1 \qquad \sum_{a_2 \in A_2} s_2(a_2) = 1$$

$$q(a_1) \geq 0, \; w(a_2) \geq 0, \; s_1(a_1) \geq 0, \; s_2(a_2) \geq 0 \quad \forall a_1 \in A_1, \forall a_2 \in A_2$$

$$s_1(a_1) \cdot q(a_1) = 0, \; s_1(a_1) \cdot q(a_1) = 0 \quad \forall a_1 \in A_1, \forall a_2 \in A_2$$

- REGRET

  - PLAYER'S i REGRET FOR PLAYING ACTION a_i IF OTHER AGENT ADOPTS ACTION PROFILE a_{-i}

$$\left[\max_{a_i' \in A_i} u_i(a_i', a_{-i})\right] - u_i(a_i, a_{-i})$$

- USEFULL IF OTHER PLAYER IS NOT COMPLETELY MALICIOUS

- MAX REGRET FOR PLAYING ACTION a_i

$$\max_{a_{-i} \in A_{-i}} \left(\left[\max_{a_i' \in A_i} u_i(a_i', a_{-i})\right] - u_i(a_i, a_{-i}) \right)$$

- MINMAX REGRET
  - ACTION a_i MINIMALIZING MAXIMUM REGRET

$$\arg\min_{a_i \in A_i} \max_{a_{-i} \in A_{-i}} \left(\left[\max_{a_i' \in A_i} u_i(a_i', a_{-i})\right] - u_i(a_i, a_{-i}) \right)$$

- CORRELATED EQUILIBRIUM
  - COORDINATE STRATEGIES BY COIN OR OTHER EXTERNAL SIGNAL
  - $G = (N, A, u)$ IS NORMAL FORM GAME
  - σ IS PROBABILITY DISTRIBUTION OVER JOINT PURE STRATEGY PROFILES

    $\sigma \in \Delta(A)$

  - WE SAY THAT σ IS CORRELATED EQUILIBRIUM IF FOR EVERY PLAYER i AND EVERY ACTION $a_i' \in A_i$ IT HOLDS

$$\sum_{a \in A} \sigma(a) u_i(a_i, a_{-i}) \geq \sum_{a \in A} \sigma(a) u_i(a_i', a_{-i})$$

  - FOR EVERY NASH EQUILIBRIUM THERE EXISTS CORRESPONDING CORRELATED NASH EQUILIBRIUM

  - LP

$$\sum_{a \in A} \sigma(a) u_i(a_i, a_{-i}) \geq \sum_{a \in A} \sigma(a) u_i(a_i', a_{-i}) \quad \forall i \in N, \ \forall a_i' \in A_i$$

$$\sum_{a \in A} \sigma(a) = 1 \qquad \sigma(a) \geq 0 \quad \forall a \in A$$

- STACKELBERG EQUILIBRIUM

- PUBLIC AUTHORITY HAVE TO ANNOUNCE POLICY

- LEADER
  - ANNOUNCES POLICY

- FOLOWER
  - PLAYS NASH EQUILIBRIUM WITH RESPECT TO THE COMMITMENT OF THE LEADER

- STACKELBERG EQUILIBRIUM
  - STRATEGY PROFILE THAT SATISFIES ABOVE CONDITIONS AND MAXIMIZES EXPECTED UTILITY OF LEADER

$$\underset{s \in S_i \, \forall i \in N \setminus \{1\} \, s_i \in BR_i(s_{-i})}{\text{ARG MAX}} \; u_1(s)$$

- $$\max_{s_1 \in S_1} \sum_{a_1 \in A_1} s_1(a_1)\, u_1(a_1, a_2)$$

$$\sum_{a_1 \in A_1} s_1(a_1)\, u_2(a_1, a_2) \geq \sum_{a_1 \in A_1} s_1(a_1)\, u_2(a_1, a_2') \quad \forall a_2' \in A_2$$

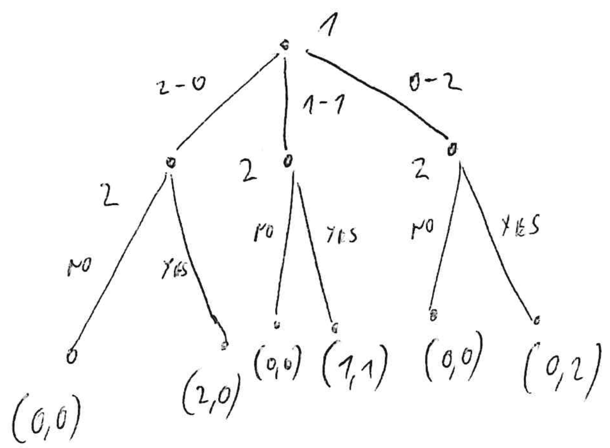
$$\sum_{a_1 \in A_1} s_1(a_1) = 1$$

- EXTENSIVE FORM REPRESENTATION

  - PLAYERS $N = \{1, 2, \dots n\}$
  - ACTIONS A
  - CHOICE NODES (HISTORIES) H
  - ACTION FUNCTION $\chi : H \to 2^A$
  - PLAYER FUNCTION $\rho : H \to N$
  - TERMINAL NODES Z
  - SUCCESSOR FUNCTION $\varphi : H \times A \to H \cup Z$
  - UTILITY FUNCTION $u = (u_1, u_2, \dots, u_n)$; $u_i : Z \to \mathbb{R}$

- PURE STRATEGY OF PLAYER i IS ASSIGNMENT OF ACTION FOR EACH STATE WHERE PLAYER i ACTS

$$S_i := \prod_{h \in H, \rho(h) = i} \chi(h)$$

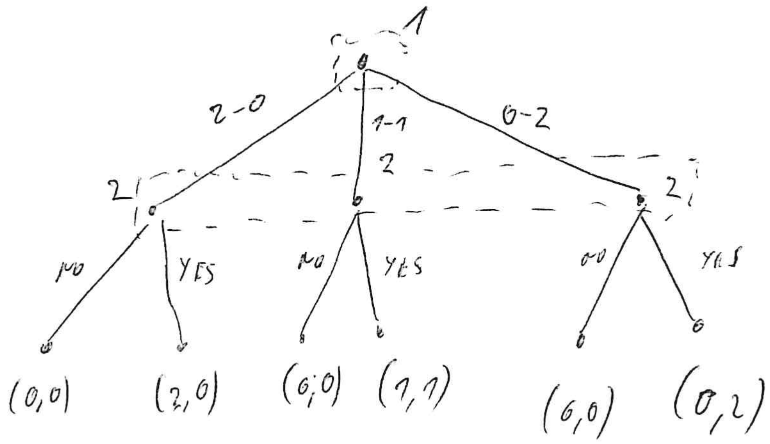


- ACTIONS:
  - $A_1 = \{2\text{-}0, 1\text{-}1, 0\text{-}2\}$
  - $A_2 = \{NO, YES\}$
- STRATEGIES
  - $S_1 = \{2\text{-}0, 1\text{-}1, 0\text{-}2\}$
  - $S_2 = \{(NO, NO, NO), (NO, NO, YES) \dots (YES, YES, YES)\}$

- NASH EQUILIBRIA
  - NOT ALL ARE COMPLETELY SEQUENTIAL IN EFG
  - THERE IS REFINEMENT FOR EFG
    - SUBGAME-PERFECT EQUILIBRIUM
  - SUBGAME
    - SUBGAME OF G ROOTED AT NODE h IS RESTRICTION OF G TO DESCENDANTS OF h
  - SUBGAME-PERFECT EQUILIBRIUM
    - ALL STRATEGY PROFILES s SUCH THAT FOR ANY SUBGAME G' OF G, THE RESTRICTION OF s TO G' IS NASH EQUILIBRIUM
    - USE MINIMAX ALGORITHM
  - EVERY EXTENSIVE-FORM GAME WITH PERFECT INFORMATION HAS AT LEAST ONE NASH EQUILIBRIUM IN PURE STRATEGIES THAT IS ALSO A SUBGAME-PERFECT EQUILIBRIUM
    - BUT NOT EVERY GAME CAN BE REPRESENTED AS EFG WITH PERFECT INFORMATION

- EFG WITH CHANCE
  - INTRODUCE NEW "PLAYER" TERMED CHANCE
  - PLAYS RANDOMIZED FIXED STRATEGY
    - PLAYERS $N = \{1, 2, \dots, n\} \cup \{c\}$
    - ACTIONS A
    - CHOICE NODES (HISTORIES) H
    - ACTION FUNCTION $\chi : H \to 2^A$
    - PLAYER FUNCTION $\rho : H \to N$
    - TERMINAL NODES Z
    - SUCCESSOR FUNCTION $\varphi : H \times A \to H \cup Z$
    - STOCHASTIC TRANSITIONS $\gamma : \Delta\{\chi^{(h)} \mid h \in H, \rho(h) = c\}$
    - UTILITY FUNCTION $u = (u_1, u_2, \dots, u_n)$; $u_i : Z \to \mathbb{R}$

- EFG WITH IMPERFECT INFORMATION
  - PLAYERS ARE NOT ABLE TO OBSERVE STATE OF GAME PERFECTLY
  - INFORMATION SET
    - STATES UNDISTINGUISHABLE TO PLAYER
  - DEFINITION
  - $G = (N, A, H, Z, X, P, \varphi, \rho, u)$ IS PERFECT-INFORMATION EFG
  - $I = (I_1, I_2, \dots, I_n)$
    - I_i IS SET OF EQUIVALENCE CLASSES ON CHOICE NODES OF PLAYER i
    - $p(h) = p(h') = i$ AND $X(h) = X(h')$ IF $h, h' \in I$ FOR SOME INFORMATION SET $I \in I_i$
    - WE CAN USE $X(I)$ INSTEAD OF $X(h)$ FOR SOME $h \in I$



- ACTIONS
  - $A_1 = \{2-0, 1-1, 0-2\}$
  - $A_2 = \{NO, YES\}$
- STRATEGIES
  - $S_1 = \{2-0, 1-1, 0-2\}$
  - $S_2 = \{NO, YES\}$
- THERE IS NO GUARANTEE THAT PURE NE EXIST IN IMPERFECT INFORMATION GAMES

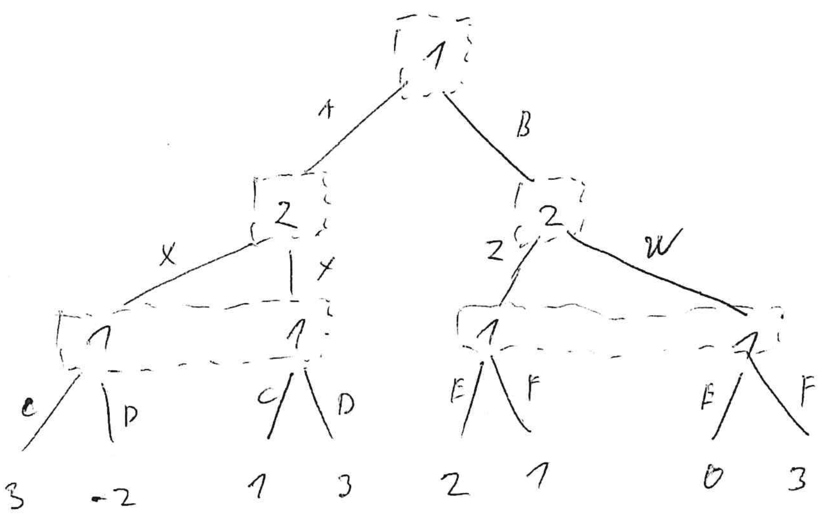
- EVERY FINITE GAME CAN BE REPRESENTED AS EFG WITH IMPERFECT INFORMATION

- BEHAVIORAL STRATEGY
    - PRODUCT OF PROBABILITY DISTRIBUTIONS OVER ACTIONS IN EACH INFORMATION SET

      $$B_i : \prod_{I \in I_i} \Delta(X(I))$$

- PERFECT - RECALL
    - AGENT KNOWS ITS HISTOR
    - IMPERFECT - RECALL EXAMPLE
        - DRUNKEN DRIVER TRYING TO EXIT MOTORWAY ON RIGH EXIT

- SEQUENCE
    - σ_i
    - ORDERED LIST OF ACTIONS OF PLAYER i EXECUTED FROM ROOT OF GAME TREE TO SOME NODE $h \in H$
    - SET OF ALL SEQUENCES OF PLAYER i
        - Σ_i



$\Delta(\Sigma_1)$	$O(\Sigma_2)$
$\emptyset$	$\emptyset$
A	X
B	Y
AC	Z
AD	W
BE	
BF	

- UTILITY FUNCTION OPERATING OVER SEQUENCES

$$g: \Sigma_1 \times \Sigma_2 \rightarrow \mathbb{R}$$

- $g(\sigma_1, \sigma_2) =$
  - $u(z)$ IFF z CORRESPONDS TO HISTORY REPRESENTED BY SEQUENCES σ_1 AND σ_2
  - 0 OTHERWISE

  - WITH CHANCE NODES
    - $\sum_{z \in Z'} C(z) \cdot u(z)$

- EXAMPLES
  - $g(\emptyset, W) = 0$
  - $g(AC, W) = 0$
  - $g(BF, W) = 3$
  - $g(A, X) = 0$

- REALIZATION PLANS
  - $r_i(\sigma_i)$ IS PROBABILITY THAT SEQUENCE σ_i WILL BE PLAYED ASSUMING PLAYER $-i$ PLAYS SUCH ACTIONS THAT ALLOW ACTIONS FROM σ_i TO BE EXECUTED

  - EXAMPLES

$r_1(\emptyset) = 1$
$r_1(A) + r_1(B) = r_1(\emptyset)$
$r_1(AC) + r_1(AD) = r_1(A)$
$r_1(BE) + r_1(BF) = r_1(B)$

$r_2(\emptyset) = 1$
$r_2(X) + r_1(Y) = r_2(\emptyset)$
$r_2(Z) + r_1(W) = r_2(\emptyset)$

- LP

$$\max_{r_1, v} \quad v(\text{root})$$

$$\text{ST} \quad r_1(\emptyset) = 1$$

$$0 \le r_1(\sigma_1) \le 1 \quad \forall \sigma_1 \in \Sigma_1$$

$$\sum_{a \in A(I_1)} r_1(\sigma_1 a) = r_1(\sigma_1) \quad \forall \sigma_1 \in \Sigma_1, \forall I_1 \in \text{inf}_1(\sigma_1)$$

$$\sum_{I' \in \text{inf}_2(\sigma_2 a)} v(I') + \sum_{\sigma_1 \in \Sigma_1} g(\sigma_1, \sigma_2 a) r_1(\sigma_1) \ge v(I) \quad \forall I \in \mathcal{I}_2, \sigma_2 = \text{SEQ}_2(I), \forall a \in A(I)$$

- APPROXIMATE ALGORITHMS
  - LEARN BEST STRATEGY VIA REPEATED PLAY
  - IDEA
    - CONSTRUCT WHOLE TREE
    - IN EACH ITERATION TRAVERSE THROUGH GAME TREE AND ADAPT STRATEGY IN EACH INFORMATION SET ACCORDING TO LEARNING RULE
    - LEARNING RULE MINIMIZES REGRET
    - ALGORITHM MINIMIZES OVERALL REGRET IN GAME
    - AVERAGE STRATEGY CONVERGES TO OPTIMAL STRATEGY

- REGRET IN EFG

$$v_i(s, I) = \sum_{z \in Z_I} \pi^s_{-i}(z[I]) \, \pi^s_i(z \mid z[I]) \, u_i(z)$$

- Z_I - LEAFS REACHABLE FROM INFORMATION SET I
- $z[I]$ - HISTORY PREFIX OF z IN I
- $\pi^s_i(h)$ - IS PROBABILITY OF PLAYER i REACHING NODE h FOLOWING STRATEGY s

- FINITELY REPEATED GAMES
  - NORMAL FORM GAME IS PLAYED REPEATEDLY
  - FINITE NUMBER OF ROUNDS
  - WE CAN SOLVE IT BY BACKWARD INDUCTION

- INFINITELY REPEATED GAMES
  - INFINITE ROUNDS
  - WE CANT USE EXTENSIVE FORM
    - THERE ARE NO LEAFS FOR VALUE ASSIGNMENT
  - AVERAGE REWARD$_i$ FOR SEQUENCE OF PAYOFFS

$$\lim_{k \to \infty} \frac{\sum_{j=1}^{k} r_i^{(j)}}{k}$$

  - DISCOUNTED REWARD

$$\sum_{j=1}^{\infty} \beta^j r_i^{(j)}$$

  - ENFORCEABLE PAYOFF PROFILE

$$r = (r_1, r_2, \dots, r_n) \quad \text{IF } \forall i \in N,\ r_i \geq v_i$$

    - WHERE v_i IS MINIMAX VALUE FOR PLAYER i

$$v_i = \min_{s_{-i} \in S_{-i}} \max_{s_i \in S_i} u_i(s_{-i}, s_i)$$

  - FEASIBLE PAYOFF PROFILE
    - IF THERE EXIST RATIONAL NONNEGATIVE VALUES α_a SUTCH THAT
    - FOR ALL i WE CAN EXPRESS r_i AS $\sum_{a \in A} \alpha_a u_i(a)$ WITH $\sum_{a \in A} \alpha_a = 1$

- FOLK THEOREM
  - n PLAYER NORMAL FORM GAME G
  - PAYOFF PROFILE $r = (r_1, r_2, \dots, r_n)$
    - IF r IS PAYOFF PROFILE FOR ANY NASH EQUILIBRIUM s OF INFINITELY REPEATED G WITH AVERAGE REWARDS, THEN FOR EACH PLAYER i, r_i IS ENFORCEABLE
    - IF r IS BOTH FEASIBLE AND ENFORCABLE, THEN r IS THE PAYOFF PROFILE FOR SOME NASH EQUILIBRIUM OF THE INFINITELY REPEATED G WITH AVERAGE REWARDS

- STOCHASTIC GAME
  - WE CAN CHANGE THE GAME MID-PLAY
  - IT IS TOUPLE (Q, N, A, P, R)
    - Q - FINITE SET OF GAMES
    - N - FINITE SET OF PLAYERS
    - A - FINITE SET OF ACTIONS
    - P - TRANSITION FUNCTION $P: Q \times A \times Q \rightarrow [0, 1]$
      - $P(q, a, q')$ IS PROBABILITY OF REACHING GAME q' AFTER JOINT ACTION a IS PLAYED IN q
    - R - REWARD FUNCTION
      - $r_i : Q \times A \rightarrow \mathbb{R}$
      - TYPES
        - DISCOUNTED
        - AVERAGE
        - AGENT WANTS TO REACH SOME GAME

- HISTORY
  - $h_t = (q_0, a_0, q_1, a_1, \dots a_{t-1}, q_t)$ DENOTES HISTORY OF t STAGES
  - H_t IS SET OF ALL POSSIBLE HISTORIES OF LENGTH t

- BEHAVIORAL STRATEGY
  - $s_i(h_t, a_{ij})$
    - PROBABILITY OF PLAYING ACTION a_{ij} FOR HISTORY h_t

- MARKOV STRATEGY
  - TYPE OF BEHAVIORAL STRATEGY
  - $s_i(h_t, a_{ij}) = s_i(h_t', a_{ij})$ IF $q_t = q_t'$ WHERE q_t IS FINAL GAME OF h_t
  - "DOESN'T CARE ABOUT HISTORY"

- STATIONARY STRATEGY
  - TYPE OF MARKOV STRATEGY
  - $s_i(h_{t_1}, a_{ij}) = s_i(h_{t_2}, a_{ij})$ IF $q_t = q_t'$ WHERE
    - q_t AND q_t' ARE FINAL GAMES OF h_{t_1} AND h_{t_2} RESPECTIVELY

- MARKOV PERFECT EQUILIBRIUM
  - STRATEGY PROFILE WHICH
  - CONSISTS OF ONLY MARKOV STRATEGIES

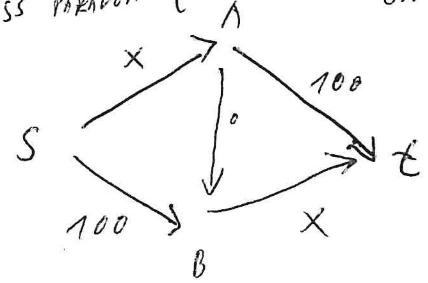
- EVERY n-PLAYER, GENERAL-SUM, DISCOUNTED-REWARD STOCHASTIC GAME HAS MARKOV PERFECT EQUILIBRIUM
  - CAN BE FOUND BY VALUE ITERATION

- SUCCINCT REPRESENTATIONS

  - COMPACT REPRESENTATION OF GAME WITH $n = |N|$ PLAYERS

  - WE WANT TO REDUCE INPUT FROM $|S|^{|n|}$ TO $|S|^{d}$ WHERE $d << |N|$

  - BRAESS PARADOX (CONGESTION GAMES)



    - EVERY ATOMIC CONGESTION GAME HAS PURE NASH EQUILIBRIUM
      - FINDABLE BY
        - PLAYERS ITERATIVELY SWITCH TO THEIR PURE BEST RESPONSE

- COOPERATIVE GAME THEORY

  - PAYOFFS GO TO COALITION WHICH REDISTRIBUTE THEM AMONG MEMBERS
  - AGENTS CHOOSE COALITION AND AGREE ON PAYOFF DISTRIBUTION
  - ~~VOK~~ BUT PLAYERS ARE SELF-INTERESTED

  - ASSUMPTION IS THAT COALITION CAN ACHIEVE MORE THAN SUM OF INDIVIDUAL AGENTS

  - ALSO CALLED COALITION GAME THEORY

  - FLOW OF GAME
    - AGENTS ANALYZE WHICH COALITIONS AND WHICH PAYOFF DISTRIBUTIONS WOULD BE BENEFICIAL FOR THEM
    - AGENTS AGREE ON COALITIONS AND PAYOFF DISTRIBUTION
    - TASK IS EXECUTED AND PAYOFF DISTRIBUTED

- COALITION GAMES
    - TRANSFERABLE UTILITY GAMES
        - PAYOFFS ARE GIVEN TO GROUPS AND DISTRIBUTED AMONG MEMBERS
        - THERE MUST BE UNIVERSAL CURRENCY THAT IS USED FOR EXCHANGE IN SYSTEM
    - NON TRANSFERABLE UTILITY GAMES
        - GROUP ACTIONS RESULTS IN PAYOFFS TO INDIVIDUAL GROUP MEMBERS
        - THERE IS NO UNIVERSAL CURRENCY

- GAME WITH TRANSFERABLE UTILITY
    - (N, v)
        - N IS FINITE SET OF PLAYERS
        - $v : 2^N \to \mathbb{R}$
            - CHARACTERISTIC FUNCTION (VALUATION FUNCTION)
            - ASSOCIATES REAL VALUED PAYOFF $v(S)$ WITH EACH COALITION $S \subseteq N$
            - WE ASSUME $v(\emptyset) = 0$

- OUTCOME AND PAY OFF VECTOR
    - OUTCOM OF GAME (N, v) IS PAIR $(CS, \vec{X})$ WHERE
        - $CS = (C_1, \dots, C_k)$
        - $\bigcup_i C_i = N$
        - $C_i \cap C_j = \emptyset$ FOR $i \neq j$
        - CS IS COALITION STRUCTURE
        - PARTITON OF N INTO COALITIONS
        - $\vec{X} = (x_1, \dots, x_n)$
        - $x_i \geq 0$ FOR ALL $i \in N$
        - $\sum_{i \in C} x_i = v(C)$ FOR EACH $C \in CS$

- IS PAYOFF (DISTRIBUTION) VECTOR
  - DISTRIBUTES VALUE OF EACH COALITION TO COALITIONS MEMBERS
  - PAYOFF IS INDIVIDUALLY RATIONAL IF
    - $x_i \geq v(\{a_i\})$

- FAIRNESS
  - SHOULD SATISFY
    - SYMMETRY
      - IF TWO AGENTS CONTRIBUTE THE SAME, THEY SHOULD GET SAME PART OF PAYOFF
    - DUMMY PLAYER
      - AGENT WHICH DO NOT ADD VALUE TO ANY COALITION SHOULD GET WHAT THEY EARN ON THEIR OWN
    - ADDITIVITY
      - IF TWO GAMES ARE COMBINED, VALUE FOR PLAYER SHOULD BE SUM OF VALUES IT GETS IN INDIVIDUAL GAMES

- SHAPLEY VALUE
  - GIVEN COALITION GAME (N, v)
  - THERE IS UNIQUE PAYOFF DIVISION $\vec{\phi}(N, v)$
  - THAT DIVIDES FULL PAYOFF OF GRAND COALITION
  - AND SATISFIES SYMMETRY, DUMMY PLAYER AND ADDITIVITY AXIOMS
  - SHAPLEY VALUE OF PLAYER i

$$\phi_i(N, v) = \frac{1}{N!} \sum_{S \subseteq N \setminus i} |S|! \, (|N| - |S| - 1)! \, [v(S \cup \{i\}) - v(S)]$$

- SUPERADDITIVE GAMES
  - $v(C \cup D) \geq v(C) + v(D)$ FOR EVERY PAIR OF DISJOINT COALITIONS $C, D \subseteq N$
  - " TWO COALITIONS CAN MERGE WITHOUT LOSING MONEY "
  - SO WE CAN ASSUME THAT PLAYERS FORM GRAND COALITION

- CONVEX GAMES
  - $v(C \cup D) \geq v(C) + v(D) - v(C \cap D)$ FOR EVERY PAIR OF COALITIONS $C, D \subseteq N$
  - CONVEX IS STRONGER THAN SUPERADITIVITY
  - " PLAYER IS MORE USEFUL WHEN HE JOINS BIGGER COALITION "

- SIMPLE GAMES
  - $v(C) \in \{0, 1\}$ FOR ANY $C \subseteq N$
  - AND v IS MONOTONE
    - $v(C) = 1$ AND $C \subseteq D$ THEN $v(D) = 1$
  - " MODEL OF YES/NO VOTING SYSTEMS "
  - VETO PLAYER
    - $v(C) = 0$ FOR ANY $C \subseteq N \setminus \{i\}$
    - BY MONOTONICITY $v(N \setminus \{i\}) = 0$

- FAIRNESS
    - HOW WELL PAYOFFS REFLECT EACH AGENT'S CONTRIBUTION
- STABILITY
    - HOW MOTIVATED IS AGENT TO STAY IN COALITION

- CORE
    - UNDER WHICH PAYMENT DISTRIBUTION IS OUTCOME OF GAME STABLE?
        - EACH SUBCOALITION MUST EARN AT LEAST THE SAME AS AT THEIR OWN
        - THIS IS CASE IF PAYOFF VECTOR IS DRAWN FROM CORE
    - $\forall S \subseteq N, \quad \sum_{i \in S} x_i \geq v(S)$
    - EQUIVALENT TO STRONG NASH EQUILIBRIUM
    - CORE EXISTS ONLY FOR CERTAIN SUBCLASS OF GAMES
    - IT IS ALSO NOT UNIQUE
    - CONVEX GAMES ALWAYS HAVE NON-EMPTY CORE
    - SIMPLE GAMES HAS NON-EMPTY CORE IFF IT HAS VETO PLAYER

- ε-CORE
    - APPROXIMATELY STABLE OUTCOME
    - $\forall S \subseteq N \quad \sum_{i \in S} x_i \geq v(S) - \varepsilon$

- LEAST CORE
    - MINIMIZES WORST-CASE DEFICIT
    - MINIMAL ε OF ALL POSSIBLE ε-CORES

- COMPACT REPRESENTATION OF COALITION GAMES

  - NAIVE REPRESENTATION IS EXPONENTIAL IN NUMBER OF PLAYERS

    - $\{1,2,3\}$ PLAYERS

      $(1)=5$ $\quad$ $(1,3)=10$

      $(2)=5$ $\quad$ $(2,3)=20$

      $(3)=5$ $\quad$ $(1,2,3)=25$

      $(1,2)=10$

  - COMPLETENESS VS SUCCINCTNESS

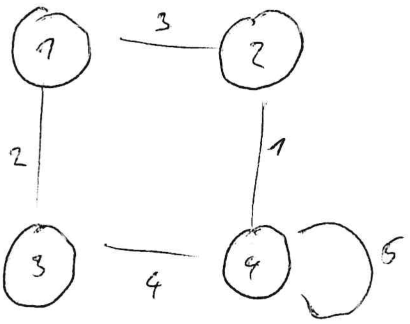
    - COMPLETE : CAN REPRESENT ANY GAME

    - SUCCINCT : SMALL SIZE BUT INCOMPLETE

- INDUCED SUBGRAPH GAMES

  - WEIGHTED GRAPH

  - VALUE OF COALITION $S \subseteq N$: $v(S) = \sum_{\{i,j\} \subseteq S} w_{ij}$



$v(\{1,2,3\}) = 3+2=5$

$v(\{4\}) = 5$

$v(\{2,4\}) = 1+5=6$

$v(\{1,3\}) = 2$

  - IT IS INCOMPLETE

  - ALL EDGES ARE NON-NEGATIVE

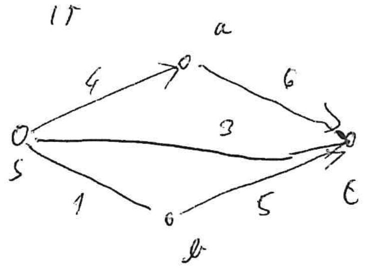
    - GAME IS CONVEX

      - CORE IS NOT EMPTY

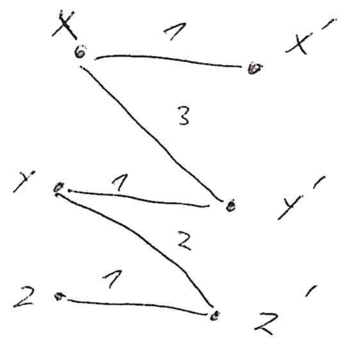
  - SHAPLEY VALUE IS EASY TO COMPUTE

    $sh_i = \frac{1}{2} \sum_{i \neq j} w_{i,j}$

- NETWORK FLOW REPRESENTATION
  - SOURCE s
  - SINK t
  - VALUE OF COALITION IS AMMOUNT OF FLOW GOING THROUGH IT



- ASSIGNMENT GAMES
  - PLAYERS ARE VERTICES OF BIPARTITE GRAPH
  - VALUE OF COALITION IS WEIGHT OF MAX-WEIGHT INDUCED MATCHING



- MATCHING GAMES
  - GENERALIZATION OF ASSIGNMENT GAMES TO OTHER THAN BIPARTITE GRAPHS

- COALITION STRUCTURE GENERATION PROBLEM
  - WE WANT COALITION STRUCTURE (ASSIGNMENT OF ALL PLAYERS TO COALITIONS) THAT MAXIMIZE SUM OF UTILITIES
  - THE NUMBER OF POSSIBLE COALITION STRUCTURES IS HUGE

$$\alpha n^{\frac{n}{2}} \leq B_n \leq n^n$$

↖ BELL NUMBER

- DYNAMIC PROGRAMMING
    - TO FIND OPTIMAL COALITION STRUCTURE
        - SPLIT SET OF ~~TWO~~ AGENTS INTO TWO SETS IN ALL POSSIBLE WAYS
        - TRY TO FIND OPTIMAL COALITION STRUCTURE IN BOTH ~~HALVES~~ HALFS
    - $$f(C) = \begin{cases} \nu(C) & \text{IF } |C| = 1 \\ \text{MAX} \left\{ \underbrace{\nu(C)}_{\text{VALUE OF WHOLE COALITION}}, \max_{\underbrace{\{C', C''\} \in P^C}_{\text{SET OF ALL PARTITIONS OF } C}} \underbrace{f(C')}_{\text{VALUE OF FIRST HALF}} + \underbrace{f(C'')}_{\text{VALUE OF SECOND HALF}} \right\} & \text{OTHERWISE} \end{cases}$$
    - GUARANTEED TO FIND OPTIMAL SOLUTION
    - BUT MANY OPERATIONS ARE REDUNDANT
        - IMPROVED IDP ALGORITHM AVOIDS REDUNDANT OPERATIONS
            - IT IS FASTEST
            - $O(3^n)$
    - IT DOESN'T PROVIDE ANY ~~SO~~ PARTIAL SOLUTION BEFORE COMPLETION
- ANYTIME ALGORITHM
    - SOLUTION IMPROVES OVER TIME
    - SOME SOLUTION EXISTS ALL THE TIME

- SOCIAL CHOICE
  - $N = \{1, \dots, n\}$ ... AT LEAST TWO INDIVIDUALS
  - U ... FINITE UNIVERSE U OF AT LEAST TWO ALTERNATIVES
  - EACH AGENT HAS PREFERENCES IN U
    - REPRESENTED BY TRANSITIVE AND COMPLETE PREFERENCE RELATION $\succeq_i$
    - $R(U)$ ... PREFERENCE RELATIONS

  - SOCIA WELFARE FUNCTION

    $f: R(U)^n \rightarrow R(U)$

    - MUST SATISFY
      - TRANSITIVITY: $a \succeq_i b \succeq_i c$ ... $a \succeq_i c$
      - COMPLETENES
        - FOR ANY PAIR OF $a, b \in U$
          - EITHER $a \succeq_i b$ OR $b \succeq_i a$
          - OR BOTH IF INDIFFERENCE IS ALLOWED $a \sim_i b$

  - PARETO OPTIMAL SOCIAL WELFARE
    - IF $a \succ_i b$ FOR ALL $i \in N$ IMPLIES THAT $a \succ_f b$

  - INDEPENDENCE OF IRRELEVANT ALTERNATIVES
    - SOCIAL PREFERENCE BETWEEN ANY PAIR OF ALTERNATIVES ONLY DEPENDS ON INDIVIDUAL PREFERENCES RESTRICTED TO THESE TWO ALTERNATIVES
    - LET R AND R' BE TWO PROFILES
    - LET a AND b BE TWO ALTERNATIVES &
    - SUCH THAT $R|_{\{a,b\}} = R'|_{\{a,b\}}$ - "PAIRWISE COMPARISION OF a AND b IS SAME IN BOTH PROFILES"
    - IIA REQUIRES THAT a AND b ARE ALSO RANKED IDENTICALLY IN $\succ_f|_{\{a,b\}} = \succ'_f|_{\{a

- EXAMPLE
  - 7 VOTERS
  - 2 ~~EXAMPLES~~ ALTERNATIVES $\{A, B\}$
    - 3 x $A \succ B$
    - 4x $B \succ A$
    - B WINS UNDER PLURALITY RULE
  - 3 ALTERNATIVES $\{A, B, C\}$ ("WE ADDED IRRELEVANT ALTERNATIVE")
    - 3x $A \succ B \succ C$
    - 2x $B \succ A \succ C$
    - 2x $C \succ B \succ A$
    - NOW A WINS, IT IS NOT INDEPENDENT

- NON-DICTATORIAL SOCIAL WELFARE FUNCTION
  - THERE IS NO AGENT WHO CAN DICTATE STRICT RANKING NO MATTER WHICH PREFERENCES OTHERS HAVE
    - NO AGENT i SUCH THAT ALL PREFERENCE PROFILES R AND ALTERNATIVES a, b, $a \succ_i b$ IMPLIES $a \succ_f b$

- ARROW'S THEOREM
  - THERE IS NO SWF THAT IS IIA, PARETO-OPTIMAL AND NON-DICTATORSHIP AT ONCE WHENEVER $|U| \geq 3$

- SOCIAL CHOICE FUNCTION
  - $f : R(U)^n \times F(U) \to F(U)$
    - SUCH THAT $f(R, A) \subseteq A$ FOR ALL R AND A
  - "SOMETIMES IT IS ENOUGH TO IDENTIFY SOCIALLY MOST DESIRABLE ALTERNATIVES"

- VOTING RULE
  - $f : R(U)^n \to F(U)$
  - SPECIAL SOCIAL CHOICE FUNCTION
  - POSITIONAL SCORING RULES
    - $m = |U|$ ALTERNATIVES
    - $S = (S_1, \dots, S_m) \in \mathbb{R}^m$ SUCH THAT
      - $S_1 \geq \dots \geq S_m$
      - AND $S_1 > S_m$
    - BORDA'S RULE
      - $S = (m-1, m-2, \dots, 0)$
    - PLURALITY RULE
      - $S = (1, 0, \dots, 0)$
    - ANTI-PLURALITY RULE
      - $S = (1, 1, \dots, 1, 0)$
  - CONDORCET WINNER
    - ALTERNATIVE a, WHICH IS PREFERED BY MORE VOTERS ~~THAN ANOTHER~~ WHEN COMPARED TO EVERY OTHER CANDIDATE
    - UNIQUE, BUT NOT ALWAYS EXIST
    - CONDORCET EXTENSION
      - VOTING RULE THAT SELECT CONDORCET WINNER WHENEVER IT EXISTS

- COMPENDANT RULE
    - GIVE ONE POINT FOR EVERY PAIRWISE MAJORITY WIN
    - GIVE $0 < \alpha < 1$ FOR EVERY PAIRWISE TIE
    - WINNER IS ALTERNATIVE WITH HIGHER SCORE

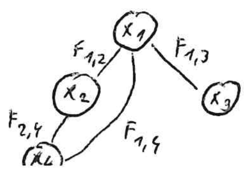
- SINGLE TRANSFERABLE VOTE
    - REMOVE ALTERNATIVES THAT ARE IN FIRST PLACE LEAST OFTEN
    - REMOVE THEM
    - REPEAT VOTING
- PAIRWISE ELIMINATION
    - PAIRWISE COMPARE ALTERNATIVES
    - THE LESS POPULAR DROPS OUT
    - REPEAT UNLESS YOU HAVE SINGLE ONE

- STRATEGIC MANIPULATION
    - RESOLUTE VOTING RULE f IS MANIPULABLE BY VOTER i
    - IF THERE EXIST PREFERENCE PROFILES R AND R'
    - SUCH THAT $R_j = R'_j$ FOR ALL $j \neq i$ AND
    - $f(R) \succ_i f(R')$
    - STRATEGY PROOF IS NOT MANIPULABLE

    - NEGATIVE ASPECTS
        - INEFFICIENT
        - UNFAIR
        - ERRATIC

    - BUT EVERY NON-IMPOSING, STRATEGY PROOF, RESOLUTE VOTING RULE IS DICTATORIAL WHEN $|U| \geq 3$

- DISTRIBUTED CONSTRAINT PROGRAMMING
  - EACH AGENT COMUNICATES WITH ITS NEIGBORS
  - CONSTRAINT NETWORK
    - TRIPLE $\langle X, D, C \rangle$
      - $X = x_1 \dots x_n$ IS SET OF VARIABLES
      - $D = \{D_1, \dots, D_n\}$ IS SET OF VARIABLE DOMAINS
        - ENUMERATES ALL POSSIBLE VALUES OF VARIABLES
      - $C = \{C_1, \dots, C_m\}$ IS SET OF CONSTRAINTS
        - C_i IS DEFINED ON SUBSET OF VARIABLES $S_i \subseteq X$
          - COMPRISE SCOPE OF CONSTRAINTS
        - HARD CONSTRAINT
          - BOOLEAN PREDICATE P_i
          - DEFINES VALID JOINT ASSIGNMENTS OF VARIABLES
          - $P_i : D_1^i \times \dots \times D_{r_i}^i \rightarrow \{F, T\}$
        - SOFT CONSTRAINT
          - FUNCTION F_i
          - MAPS EVERY POSSIBLE JOINT ASSIGMENT OF ALL VARIABLES TO REAL VALUE
          - $F_i : D_1^i \times \dots \times D_{r_i}^i \rightarrow \mathbb{R}$
  - BINARY CONSTRAINT NETWORK
    - EACH CONSTRAINT IS DEFINED OVER TWO VARIABLES
    - CAN BE REPRESENTED BY CONSTRAINT GRAPH
    - EVERY CONSTRAINT NETWORK CAN BE TRANSFORMED TO BINARY CONSTRAINT NETWORK



- CSP
  - CONSTRAINT SATISFACTORY PROBLEM
  - FIND ASSIGNMENT THAT SATISFIES ALL CONSTRAINTS

- COP
  - CONSTRAINT OPTIMIZATION PROBLEM
  - FIND ASSIGNMENT THAT SATISFIES ALL CONSTRAINTS
  - AND OPTIMIZE GLOBAL FUNCTION
    - I.E. $F = \sum F_i$

- DISTRIBUTED CONSTRAINT REASONING
  - CONSTRAINT NETWORK $\langle X, D, C \rangle$ AND SET OF AGENTS $A = A_1, \ldots, A_k$ WHERE EACH AGENT
    - CONTROLS SUBSET OF VARIABLES $X_i \subseteq X$
    - IS AWARE OF CONSTRAINTS OF VARIABLES IT CONTROLS
    - COMMUNICATES ONLY WITH ITS NEIGHBORS
  - SYNCHRONOUS
    - FEW AGENTS ARE ACTIVE
    - LOW DEGREE OF CONCURRENCY
    - ALGORITHMS ARE DIRECT EXTENSIONS OF CENTRALIZED ONES
  - ASYNCHRONOUS
    - ALL AGENTS ARE ACTIVE SIMULTANEOUSLY
    - HIGH DEGREE OF CONCURRENCY
    - MUST BE ABLE TO HANDLE OLD INFORMATIONS

- ASYNCHRONOUS BACTRACKING ALGORITHM
  - ABT
  - AGENT SENDS MESSAGES TO OTHERS DIRECTLY
  - DELAY OF MESSAGE IS FINITE BUT RANDOM
  - FOR ANY PAIR OF AGENTS, MESSAGES ARE RECIEVED IN THE ORDER THEY WERE SENT
  - AGENTS KNOWS ONLY CONSTRAINTS THEY ARE INVOLVED IN
  - EACH AGENT HAS ITS OWN VARIABLE
  - CONSTRAINTS ARE BINARY (2 VARIABLES INVOLVED)
  - IDEA
    - ALGORITHM SORTS AGENTS ~~AND~~ ACORDING TO SOME PRIORITY (MAY BE ARTIFICIAL)
    - HIGHER PRIORITY AGENT j INFORMS LOWER PRIORITY AGENT k OF ITS ASIGNMENT
    - LOWER PRIORITY AGENT k CHECKS SHARED C_{jk} CONSTRAINT WITH ITS OWN ASSIGNMENT
      - IF CONSTRAINTS SATISFIED THAN ALL IS OK, NO ACTION REQUIRED
      - OTHERWISE k CHOOSES DIFFERENT VALUE CONSISTENT WITH j
        - IS SUCH CHOICE EXISTS, AGENT k ADOPTS THIS VALUE AND INFORMS OTHER LOW PRIORTY AGENTS
        - IF NOT, AGENT k UPDATES NOGOODS AND SENDS MESSAGE TO j

- FOUR OPERATIONS
  - ASSIGNMENT
    - j TAKES VALUE v
    - j INFORMS ALL LOWER PRIORITY AGENTS IN NEIGHBORHOOD
    - OK? MESSAGE
  - BACKTRACK
    - NOGOOD
    - k HAS NO VALUE CONSISTENT WITH HIGHER-PRIORITY AGENTS
    - k SENDS NOGOOD TO LOWEST HIGHER PRIORITY AGENT IN HIS NEIGHBORHOOD
  - NEW LINKS
    - j RECIEVES NOGOOD MENTIONING i
    - BUT i IS NOT CONNECTED WITH j
    - j ASKS i TO SETUP LINK
  - STOP
    - NO SOLUTION DETECTED BY AGENT

- SOLUTION IS FOUNDED WHEN ALL AGENTS ARE SILENT FOR A WHILE
  - THIS MEANS THAT EVERY CONSTRAINT IS SATISFIED
  - SOLUTION IS DETECTED BY SPECIALIZED ALGORITHMS OUTSIDE ABT

- ADOPT
  - ASYNCHRONOUS DISTRIBUTED OPTIMIZATION
  - FOR OPTIMALLY SOLVING DCOP
  - AGENTS CHOOSING BEST VALUE BASED ON CURRENT AVAILABLE INFORMATION
  - BACKTRACK THRESHOLDS USED TO SPEED UP THE SEARCH OF PREVIOUSLY EXPLORED SOLUTIONS
  - OPPORTUNISTIC BEST-FIRST SEARCH STRATEGY
    - AGENT KEEPS ON CHOOSING VALUE WITH MINIMUM LOWER BOUND
    - LOWER BOUND IS MORE SUITABLE FOR ASYNCHRONOUS SEARCH
      - WE DON'T HAVE TO HAVE GLOBAL COST INFORMATION
  - AGENT KEEPS LOWER AND UPPER BOUND ON COST FOR SUBPROBLEM BELOW IT AND ON a SUB-PROBLEM FOR EACH OF ITS CHILD
  - IT TELLS TO CHILDREN TO LOOK FOR SOLUTION, BUT IGNORE ANY PARTIAL SOLUTION WHOSE COST IS ABOVE LOWER BOUND
  - AGENTS MUST BE ARRANGED IN DEPTH-FIRST SEARCH TREE
  - MESSAGES
    - VALUE
      - PARENT INFORMS ~~[illegible]~~ DESCENDANTS IT HAS TAKEN VALUE v
    - COST
      - CHILD INFORMS PARENT OF BEST COST OF ITS ASSIGNMENT
    - THRESHOLD
      - PARENT → CHILDREN
      - MINIMUM COST IN CHILD IS AT LEAST THRESHOLD
    - TERMINATION
      - PARENT → CHILDREN
      - ~~[illegible]~~ SOLUTION FOUND, STOP

- BEST FIRST SEARCH

$$OPT_{x_j}(c) = \min_{d \in d_j} \left(\delta_j(d) + \sum_{x_k \in children(x_j)} OPT_{x_k}(c \cup \{x_j, d\}) \right)$$

- BEST VALUE OF x_j IS VALUE MINIMIZING SUM OF x_j's LOCAL COST AND LOWEST COST OF CHILDREN UNDER THE CONTEXT EXTENDED WITH THE ASSIGNMENT
- OPT_{x_k} ARE INCREMENTALLY BOUNDED USING $[lb_k, ub_k]$

- APROXIMATE ALGORITHMS
  - SACRIFISE OPTIMALITY FOR COMPUTATIONAL DEMANDS
  - SUITED FOR LARGE-SCALE PROBLEMS
  - LOCAL SEARCH
    - START FROM RANDOM ASSIGNMENT
    - DO LOCAL MOVES IF NEW ASSIGNMENT IMPROVES THE VALUE
    - SEARCH STOPS IN LOCAL MINIMUM
    - APPROACHES
      - RANDOMIZE TO DECIDE WHETHER AGENT IS GOING TO ACT OR
      - NEGOTIATE WITH NEIGHBORS
        - AGENTS COMPUTE AND EXCHANGE POSSIBLE GAINS AND ONLY ONE WITH MAXIMUM GAIN EXECUTES ACTION

- FRODO
  - FRAMEWORK FOR OPEN/DISTRIBUTED OPTIMIZATION
    - FOR EXPERIMENTAL EVALUATION OF DCSP AND DCOP ALGORITHMS.

- AUCTIONS AND RESOURCE ALLOCATIONS
    - AUCTION
        - PROTOCOL THAT ALLOWS AGENTS TO INDICATE THEIR INTEREST IN ONE OR MORE RESOURCES
        - IT DETERMINE ALLOCATION OF RESOURCES AND SET OF PAYMENTS BY AGENTS
        - EXAMPLE OF MECHANISM DESIGN
    - INDICATORS OF SOCIAL WELFARE
        - EFFICIENCY
            - PARETO OPTIMALITY
                - AGREEMENT BE SUCH THAT THERE IS NO ALTERNATIVE AGREEMENT THAT WOULD BE BETTER FOR SOME AND NOT WORSE FOR ANY OTHER
            - UTILITARIANISM
                - SUM OF PAYOFFS SHOULD BE AS HIGH AS POSSIBLE
                - SUM OF INDIVIDUAL UTILITIES
        - FAIRNESS
            - ENVY-FREE
                - NO AGENT PREFER TO TAKE BUNDLE ALOCATED TO ONE OF ITS PEERS RATHER THAN KEEPING THEIR OWN
            - EGALITARIANISM
                - AGENT THAT IS GOING TO BE WORST OFF SHOULD BE AS WELL OFF AS POSSIBLE
                - MINIMUM OF INDIVIDUAL UTILITIES

- SINGLE-ITEM ONE-SIDED AUCTION
    - ENGLISH
        - STANDARD AUCTION
        - STARTS AT LOW PRICE
        - AGENTS CAN INCREASE PRICE
    - JAPANESE
        - LOW PRICE
        - IT IS INCREASING
        - AGENTS HAVE TO CONFIRM THEY STILL WANT TO BUY AT THAT PRICE
    - DUTCH
        - HIGH PRICE
        - IT IS DROPING
        - BUYS FIRST AGENT, WHICH ANNOUNCES THE BUY
    - 1-st PRICE SEALED-BID
        - AGENTS SEACRETLY ANNOUNCE THE PRICE
        - HIGHEST ONE BUYS
    - 2-ND PRICE SEALED-BID
        - WINNING AGENT BUYS FOR SECOND HIGHEST PRICE

- BAYESIAN GAME
  - TOUPLE $\langle N, A, \Theta, p, u \rangle$
    - $N = \{1, \dots, n\}$ IS SET OF PLAYERS
    - $\Theta = \Theta_1 \times \dots \times \Theta_n$
      - Θ_i IS TYPE SPACE OF PLAYER i
    - $A = A_1 \times \dots \times A_n$
      - A_i IS SET OF ACTIONS FOR PLAYER i
    - $p: \Theta \rightarrow [0, 1]$ IS COMMON PRIOR OVER TYPES
    - $u = u_1 \dots u_n$
      - $u_i: \Theta \rightarrow \mathbb{R}$ IS UTILITY FUNCTION OF PLAYER i

- BAYES-NASH EQUILIBRIUM
  - RATIONAL RISK NEUTRAL PLAYERS ARE SEEKING TO MAXIMIZE THEIR EXPECTED PAYOFF GIVEN THEIR BELIEFS ABOUT THE OTHER PLAYER'S TYPES

- AUCTION AS BAYESIAN GAME
  - PLAYERS ACTION CORRESPONDS TO HIS BID b_i
  - PLAYER TYPES θ_i CORRESPOND TO PLAYERS PRIVATE VALUATIONS v_i OVER AUCTION ITEMS
  - PAYOFF OF PLAYER i CORRESPONDS TO HIS VALUATION OF ITEM v_i MINUS PAYED BID b_i

- WE WANT TO FIND MECHANISM WITH TRUTH TELLING AS DOMINANT STRATEGY
  - $b_i = v_i$
  - IT HOLDS FOR SECON PRICE SEALED BID ASSUMING
    - INDEPENDENT PRIVATE VALUES
    - RISK NEUTRAL PLAYERS

- DUTCH AND FIRST SEALED BID AUCTIONS
  - STRATEGICALLY EQUIVALENT
  - NO DOMINANT STRATEGY
  - AGENT MUST TRADE-OFF BETWEEN PROBABILITY OF WINNING VS. AMOUNT PAID UPON WINNING
  - OPTIMAL STRATEGY DEPENDS ON ASSUMPTIONS ABOUT OTHERS VALUATIONS
  - EQUILIBRIA FOR TWO RISK-NEUTRAL PLAYERS WHOSE VALUATIONS ARE DRAWN INDEPENDENTLY AND UNIFORMLY FROM INTERVAL $[0,1]$
    - $\left(\frac{v_1}{2}, \frac{v_2}{2}\right)$
    - n-PLAYER CASE

$$\left(\frac{n-1}{n} v_1, \dots, \frac{n-1}{n} v_n\right)$$

- ENGLISH AND JAPANESE
  - COMPLICATED STRATEGY SPACE
  - EXTENSIVE FORM GAME
  - DOMINANT STRATEGY IS TO BID UP TO AGENT'S VALUATIONS

- MULTIITEM AUCTIONS
  - AUCTION FOR BUNDLES
    - $Z = \{z_1, \dots, z_m\}$ IS SET OF ITEMS TO BE AUCTIONED
    - $v_i : 2^Z \rightarrow \mathbb{R}$ INDICATES HOW MUCH A BUNDLE $Z \subseteq Z$ IS WORTH TO AGENT i
  - NORMALIZATION

$$v(\emptyset) = 0$$

  - FREE DISPOSAL

$$Z_1 \subseteq Z_2 \text{ IMPLIES } v(Z_1) \leq v(Z_2)$$

- COMBINATORIAL AUCTIONS DO NOT HAVE TO HAVE ADDITIVE VALUATION FUNCTION
  - COMPLEMENTARITY
    - "LEFT AND RIGHT SHOE"
    - $v(Z_1 \cup Z_2) > v(Z_1) + v(Z_2)$
  - SUBSTITUTABLE ITEMS
    - "~~TWO~~ TWO CINEMA TICKETS FOR THE SAME TIME BUT DIFFERENT THEATERS"
    - $v(Z_1 \cup Z_2) < v(Z_1) + v(Z_2)$
  - DIFFICULTIES
    - NP-HARD
    - REPRESENTATION AND COMMUNICATION
      - REPRESENTATION GROWS EXPONENTIALLY WITH NUMBER OF OBJECTS
- SINGLE-MINDED COMBINATORIAL AUCTIONS
  - VALUATION v IS SINGLE MINDED
  - IF THERE EXISTS BUNDLE Z^* AND VALUE $v^* \in \mathbb{R}^+$
  - SUCH THAT $v(Z) = v^*$ FOR ALL $Z^* \subseteq Z$ AND $v(Z') = 0$ FOR ALL OTHER Z'
  - SINGLE-MINDED BID IS PAIR (Z^*, v^*)